4 Conformal Mappings

4.1 Biholomorphic Mappings

Definition 4.1. Let X be a topological space and S a set. A function $f : X \to S$ is called *locally injective at* $x \in X$ iff there is a neighborhood $U \subseteq X$ of x such that f restricted to U is injective. f is called *locally injective* iff it is locally injective at each $x \in X$.

Theorem 4.2. *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$, $a \in D$ and $p := f(a)$. Suppose that $f - p$ has a zero of order m at a. Then there exist $\epsilon > 0$ and $\delta > 0$ with $\overline{B_\delta(a)} \subset D$ such that for $q \in B_\epsilon(p) \setminus \{p\}$ the function $f - q$ has exactly m distinct simple zeros for $z \in B_\delta(a)$ and $f - p$ has no further zeros in $z \in B_\delta(a)$.*

Proof. Since f is not constant (otherwise $f - p$ could not have a zero of finite order according to Proposition 3.2), neither $f - p$ nor f' are constant zero. So the zeros of both $f - p$ and f' are isolated. This implies that we can find $\delta > 0$ with $\overline{B_\delta(a)} \subset D$ such that $f(z) - p \neq 0$ and $f'(z) \neq 0$ for all $z \in \overline{B_\delta(a)} \setminus \{a\}$. Now set $\epsilon := \min_{\zeta \in \partial B_\delta(a)}\{|f(\zeta) - p|\}$. Then, if $q \in B_\epsilon(p)$,

$$|(f(\zeta) - p) - (f(\zeta) - q)| < \epsilon \leq |f(\zeta) - p| \quad \forall \zeta \in \partial B_\delta(a).$$

So, by Rouché's Theorem (Theorem 3.21), $f - p$ and $f - q$ must have the same numbers of zeros, counted with multiplicity, in $B_\delta(a)$, namely m. If $q \neq p$ these are all simple by Proposition 3.2 because $f'(z) \neq 0$ for $z \in B_\delta(a) \setminus \{a\}$. □

Proposition 4.3. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is locally injective at $a \in D$ iff $f'(a) \neq 0$. Moreover, f is locally injective in D iff f' is nowhere zero in D.*

Proof. Let $a \in D$ and $p := f(a)$. Suppose first that $f'(a) = 0$. Then, either f is constant or $f - p$ has a zero of order $m \geq 2$ at a. In the first case the lack of local injectivity is trivial. In the second case consider an open neighborhood $U \subseteq D$ of a. Applying Theorem 4.2, there exists $\epsilon > 0$ such that for $q \in B_\epsilon(p) \setminus \{p\}$ the equation $f(z) = q$ has at least two distinct solutions for $z \in U$. In particular, f is not injective in U. Since U was arbitrary, f is not locally injective at a.

Now suppose $f'(a) \neq 0$. Then, $f - p$ has a simple zero at a. Applying Theorem 4.2, there exist $\epsilon > 0$ and $\delta > 0$ with $B_\delta(a) \subset D$ such that for all $q \in B_\epsilon(p)$ the equation $f(z) = q$ has exactly one solution in $B_\delta(a)$. By continuity of f, $U := f^{-1}(B_\epsilon(p)) \cap B_\delta(a)$ is an open neighborhood of a. Clearly, f is injective in U, showing that f is locally injective at a. □

Recalling Section 1.3 we see that the concept of conformality is equivalent to holomorphicity combined with local injectivity.

Definition 4.4. Let $D, D' \subseteq \mathbb{C}$ be regions. A map $f : D \to \mathbb{C}$ with $f(D) = D'$ is called a *biholomorphic map* from D to D' iff f is holomorphic and has a holomorphic inverse $f^{-1} : D' \to \mathbb{C}$. If such a map exists, D and D' are said to be *conformally equivalent.*

Theorem 4.5. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is a biholomorphic mapping from D to $f(D)$ iff f is injective.*

Proof. Clearly, biholomorphicity implies injectivity. For the converse assume that f is injective. By continuity, the image $D' := f(D)$ is connected. Moreover, by the Open Mapping Theorem (Theorem 2.40), D' is open. So D' is a region as it cannot be empty. Since f is injective, the inverse map $f^{-1} : D' \to D$ exists. Again using the Open Mapping Theorem, f^{-1} is continuous. Moreover, by Proposition 4.3 f' is nowhere zero. Applying Proposition 1.7 we conclude that f^{-1} is everywhere complex differentiable, i.e., it is holomorphic. □

In the following $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ denotes the *upper half-plane* in $\mathbb{C}$.

<u>Exercise</u> 43. Show that $z \mapsto -z^2$ restricted to $\mathbb{H}$ is a biholomorphic mapping. Onto which region?

4.2 Conformal Automorphisms of $\mathbb{C}$ and $\mathbb{C}^\times$

Definition 4.6. Let $D \subseteq \mathbb{C}$ be a region. A biholomorphic mapping from D to D is called a *conformal automorphism* of D. The group of conformal automorphisms of D is denoted $\mathrm{Aut}(D)$.

As a first example we consider conformal automorphisms of $\mathbb{C}$. The following ones are obvious:

1. $T_a : z \mapsto z + a$ where $a \in \mathbb{C}$ is the *translation* by a.

2. $R_\theta : z \mapsto e^{i\theta} z$ where $\theta \in [0, 2\pi)$ is the *rotation* by the angle θ around the origin in positive direction.

3. $S_r : z \mapsto rz$ where $r \in \mathbb{R}^+$ is the *scaling* by the factor r around the origin.

<u>Exercise</u> 44. Show that the group generated by translations, rotations and scalings of $\mathbb{C}$ consists precisely of the biholomorphic transformations $\mathbb{C} \to \mathbb{C}$ of the form

$$z \mapsto az + b \quad \text{with} \quad a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}.$$

As we shall see soon there are in fact no further automorphisms of $\mathbb{C}$. Another interesting example is the *punctured plane* $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. In addition to the rotations and scalings already seen above, there is another elementary automorphism of $\mathbb{C}^\times$ given by

$$I : z \mapsto \frac{1}{z}, \quad \text{called } \textit{inversion}.$$

We shall see that there are no further automorphisms of $\mathbb{C}^\times$ than those generated by rotations, scalings and inversions.

Lemma 4.7. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$ be injective. Then, either a is a pole of order one or it is a removable singularity and the continuation of f to D is injective.*

Proof. Suppose that a is a removable singularity and denote the continuation of f by $\tilde{f} \in \mathcal{O}(D)$. Assume that $\tilde{f}$ is not injective. Since f is injective this means there exists $z \in D \setminus \{a\}$ such that $\tilde{f}(a) = \tilde{f}(z)$. Choose $r > 0$ such that $r < |z - a|/2$ and $B_r(a) \subseteq D$ and $B_r(z) \subseteq D$. By the Open Mapping Theorem (Theorem 2.40) $\tilde{f}(B_r(z))$ and $\tilde{f}(B_r(a))$ are open and so is their intersection $U := \tilde{f}(B_r(z)) \cap \tilde{f}(B_r(a))$. But by assumption U is not empty as it contains $f(a)$. Since U is open there exists $p \in U$ with $p \neq f(a)$. Then there must exist $z_1 \in B_r(a) \setminus \{a\}$ and $z_2 \in B_r(z)$ such that $f(z_1) = p = f(z_2)$ contradicting the injectivity of f. Thus, $\tilde{f}$ must be injective.

Suppose now that a is not a removable singularity. Let $r > 0$ such that $\overline{B_r(a)} \subset D$ and define $D' := D \setminus \overline{B_r(a)}$. By the Open Mapping Theorem (Theorem 2.40) the sets $f(D')$ and $f(B_r(a) \setminus \{a\})$ are both open and non-empty, but their intersection is empty by injectivity. Thus, $f(B_r(a) \setminus \{a\})$ cannot be dense in $\mathbb{C}$. By the Casorati-Weierstrass Theorem (Theorem 3.10) this implies that a is not an essential singularity. Hence, it must be a pole. This implies that there is $s > 0$ such that $B_s(a) \subseteq D$ and $f(z) \neq 0$ for all $z \in B_s(a) \setminus \{a\}$. Define $g \in \mathcal{O}(B_s(a) \setminus \{a\})$ by $g(z) := 1/f(z)$. Note that g is injective since f is. Also, a is a pole of f, so a is a removable singularity of g. This implies by the above part of the proof that the continuation $g \in \mathcal{O}(B_s(a))$ is still injective. In particular, g is locally injective at a, so Proposition 4.3 implies that $g'(a) \neq 0$. On the other hand $g(a) = 0$, so a is a zero of order one of g, implying that it is a pole of order one of f. □

Theorem 4.8. *Every injective holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is an automorphism of $\mathbb{C}$ and can be written in the form*

$$z \mapsto az + b \quad \textit{for some} \quad a \in \mathbb{C}^\times, b \in \mathbb{C}.$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be the power series expansion of f. Define the function $g \in \mathcal{O}(\mathbb{C}^\times)$ by $g(z) := f(1/z)$. Then, g is injective and has the Laurent series expansion

$$g(z) = \sum_{n=0}^{\infty} c_n z^{-n}$$

in $A_{0,\infty}(0)$. By Lemma 4.7, 0 is either a removable singularity of g or a pole of order one. This implies $c_n = 0$ for all $n \geq 2$ by Proposition 3.17. By injectivity $c_1 \neq 0$, so f has the stated form and is an automorphism of $\mathbb{C}$. □

Corollary 4.9. $\mathbb{C}$ *is not conformally equivalent to any proper subset.*

Theorem 4.10. *Every injective holomorphic mapping $f : \mathbb{C}^\times \to \mathbb{C}^\times$ is an automorphism of $\mathbb{C}^\times$ and takes either the form*

$$z \mapsto az \quad \textit{or} \quad z \mapsto \frac{a}{z} \quad \textit{for some} \quad a \in \mathbb{C}^\times.$$

Proof. According to Lemma 4.7, 0 can either be a removable singularity of f or a pole of order one. In the first case, the continuation $\tilde{f} \in \mathcal{O}(\mathbb{C})$ is injective by the same Lemma. Thus, $\tilde{f}$ is automorphism of $\mathbb{C}$ and $\tilde{f}(z) = az + b$ for some $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$ by Theorem 4.8. But must have $\tilde{f}^{-1}(\{0\}) \neq \emptyset$ while $f^{-1}(\{0\}) = \emptyset$, implying $\tilde{f}(0) = 0$. Thus, $b = 0$. In the second case define the injective holomorphic function $g : \mathbb{C}^\times \to \mathbb{C}^\times$ by $g(z) := 1/f(z)$. Since f has a pole at 0, g has a removable singularity at 0. So we can apply the first part of the proof to g showing that $g(z) = \tilde{a}z$ for some $\tilde{a} \in \mathbb{C}^\times$. Setting $a := 1/\tilde{a}$ we find $f(z) = a/z$, completing the proof. □

Exercise 45. Show that $\mathbb{C}^\times$ is conformally equivalent to $\mathbb{C} \setminus \{p\}$ for any $p \in \mathbb{C}$, but not to any other subset of $\mathbb{C}$.

4.3 Conformal Automorphisms of $\mathbb{D}$

We now consider the conformal automorphisms of the open unit disk $\mathbb{D} := B_1(0)$. Among the transformations we have seen so far, the rotation by an angle θ around the origin is obviously an automorphism of $\mathbb{D}$. A less obvious automorphism is given by

$$D_w : z \mapsto \frac{z-w}{\overline{w}z-1}, \quad \text{where} \quad w \in \mathbb{D}.$$

Exercise 46. Verify the following properties of the transformation D_w: (a) it is an automorphism of $\mathbb{D}$, (b) it is self-inverse, i.e., composing the transformation with itself yields the identity on $\mathbb{D}$, (c) it interchanges the points 0 and w.

We shall see that the group generated by rotations R_θ and by transformations D_w is already the full automorphism group of $\mathbb{D}$.

Lemma 4.11 (Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Then,*

$$|f(z)| \le |z| \quad \forall z \in \mathbb{D} \quad \textit{and} \quad |f'(0)| \le 1.$$

Moreover, if $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then there is $a \in \mathbb{C}$ with $|a| = 1$ such that $f(z) = az$ for all $z \in \mathbb{D}$.

Proof. Since f has a zero at 0, there is $g \in \mathcal{O}(\mathbb{D})$ such that $f(z) = zg(z)$ and moreover, $f'(0) = g(0)$. Since $|f(z)| < 1$ for all $z \in \mathbb{D}$, we have for any $0 < r < 1$,

$$\|g\|_{\partial B_r(0)} < \frac{1}{r}.$$

On the other hand, applying Proposition 2.34 to $B_r(0)$ we have

$$|g(z)| \le \|g\|_{\partial B_r(0)} < \frac{1}{r} \quad \forall z \in B_r(0).$$

Since r can be chosen arbitrarily close to 1, we get, for all $z \in \mathbb{D}$, $|g(z)| \le 1$. This translates to the first stated inequality if $z \ne 0$ and to the second stated inequality if $z = 0$. If either $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then $|g(z)| = 1$ for some $z \in \mathbb{D}$. Then, by Theorem 2.33, g is constant, i.e, there is $a \in \mathbb{C}$ such that $g(z) = a$ for all $z \in \mathbb{D}$. Consequently, $f(z) = az$. Observe also that $|a| = 1$. □

Proposition 4.12. *Let $f : \mathbb{D} \to \mathbb{D}$ be biholomorphic and $f(0) = 0$. Then, f is a rotation, i.e., there exists $\theta \in [0, 2\pi)$ such that $f = R_\theta$.*

Proof. Applying Lemma 4.11 to both f and f^{-1} yields,

$$|f(z)| \le |z| \quad \text{and} \quad |f^{-1}(z)| \le |z| \quad \forall z \in \mathbb{D}.$$

Replacing z by $f(z)$ in the second inequality yields, $|z| \le |f(z)|$ for all $z \in \mathbb{D}$. Thus, we actually find $|f(z)| = |z|$ for all $z \in \mathbb{D}$. By Lemma 4.11 this implies that there exists $a \in \mathbb{C}$ with $|a| = 1$ and $f(z) = az$, i.e., f is a rotation. $\square$

Theorem 4.13. *The group of automorphisms of $\mathbb{D}$ is generated by rotations R_θ and transformations D_w. In particular, any automorphism of $\mathbb{D}$ can be written uniquely as a composition $R_\theta \circ D_w$ for some $\theta \in [0, 2\pi)$ and some $w \in \mathbb{C}$.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$. Set $w := f^{-1}(0)$ and define $g := f \circ D_w$. Then $g \in \mathrm{Aut}(\mathbb{D})$ with the property that $g(0) = 0$. Applying Proposition 4.12 to g yields that g is a rotation. That is, there exists $\theta \in [0, 2\pi)$ such that $g = R_\theta$. Then, $f = R_\theta \circ D_w$, since $D_w \circ D_w = \mathrm{id}$. To see uniqueness suppose that also $f = R_{\theta'} \circ D_{w'}$. Then $f^{-1}(0) = (R_{\theta'} \circ D_{w'})^{-1}(0) = D_{w'}^{-1}(0) = w'$, so $w' = w$. But composing with D_w yields then $R_{\theta'} = R_\theta$ which implies $\theta' = \theta$. $\square$

<u>Exercise</u> 47. Show that the set of automorphisms of $\mathbb{D}$ is identical to the set of transformations $\mathbb{D} \to \mathbb{D}$ of the form

$$z \mapsto \frac{xz + y}{\overline{y}z + \overline{x}} \quad \text{with} \quad x, y \in \mathbb{C} \quad \text{and} \quad |x| > |y|.$$

<u>Exercise</u> 48. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $a \in \mathbb{D}$ such that $f(a) = 0$. Show that

$$|f(z)| \le \frac{|z - a|}{|\overline{a}z - 1|} \quad \forall z \in \mathbb{D}.$$

Moreover, in case of equality for some $z \in \mathbb{D} \setminus \{a\}$, f is automorphism of $\mathbb{D}$.

4.4 Möbius Transformations

It turns out that all the biholomorphic transformations we have considered so far can be written as rational maps that arise as quotients of polynomials of degree one. It turns out that maps of this type are always biholomorphic and permit the understanding of a variety of conformal equivalences and automorphism groups.

To each complex matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$ or $d \neq 0$ we associate the rational function $M_A \in \mathcal{M}(\mathbb{C})$ given by

$$M_A(z) := \frac{az+b}{cz+d}.$$

Since

$$M'_A(z) = \frac{\det A}{(cz+d)^2}$$

we see that M_A is constant if $\det A = 0$. In the following we shall restrict to the case $\det A \neq 0$. M_A is then called a *Möbius transformation* or *fractional linear transformation*. We denote the set of these meromorphic functions by **Möb**. Recall that $\mathrm{GL}_2(\mathbb{C})$, the group of *general linear transformations* in $\mathbb{C}^2$, is the group of complex 2×2-matrices with non-zero determinant.

Proposition 4.14. *The set of Möbius transformations* ***Möb*** *forms a group by composition. Moreover, the map* $\mathrm{GL}_2(\mathbb{C}) \to$ ***Möb*** *given by* $A \mapsto M_A$ *is a group homomorphism, i.e., we have*

$$M_{AB} = M_A \circ M_B \quad \forall A, B \in \mathrm{GL}_2(\mathbb{C}).$$

Proof. **Exercise.** □

Exercise 49. Verify that the upper triangular matrices (with non-vanishing determinant) form a subgroup of $\mathrm{GL}_2(\mathbb{C})$. Show that the image of this subgroup under the map $\mathrm{GL}_2(\mathbb{C}) \to$ **Möb** is the group $\mathrm{Aut}(\mathbb{C})$. Identify the upper triangular matrices corresponding to translations, rotations and reflections.

Exercise 50. Verify that the other Möbius transformations also define biholomorphic mappings. Between which regions?

Recall that $\mathrm{GL}_2^+(\mathbb{R})$ is the group of *orientation-preserving general linear transformations* of $\mathbb{R}^2$, i.e., these are 2×2-matrices with real entries and positive determinant.

Proposition 4.15. *The restriction of the map* $\mathrm{GL}_2(\mathbb{C}) \to$ ***Möb*** *to the subgroup* $\mathrm{GL}_2^+(\mathbb{R})$ *yields Möbius transformations that are conformal automorphisms of* $\mathbb{H}$*. That is, we obtain a group homomorphism* $\mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$.

Proof. **Exercise.** □

Proposition 4.16. *Let* $D, D' \subseteq \mathbb{C}$ *be regions such that* D *and* D' *are conformally equivalent. Then* $\mathrm{Aut}(D)$ *and* $\mathrm{Aut}(D')$ *are isomorphic. In particular, every biholomorphic mapping* $D \to D'$ *yields such an isomorphism.*

Proof. Let $f : D \to D'$ be a biholomorphic mapping. Then, an isomorphism $\mathrm{Aut}(D) \to \mathrm{Aut}(D')$ is given by $g \mapsto f \circ g \circ f^{-1}$. $\square$

Exercise 51. Show that the *Cayley map* $M_C \in \mathcal{M}(\mathbb{C})$ given by

$$C := \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}$$

is a biholomorphic map from $\mathbb{H}$ to $\mathbb{D}$.

Proposition 4.17. *Consider the group homomorphism* $\mathrm{GL}^+(\mathbb{R}) \to \mathrm{Aut}(\mathbb{D})$ *given by* $A \mapsto M_C \circ M_A \circ M_C^{-1}$ *induced by the Cayley map* $M_C : \mathbb{H} \to \mathbb{D}$. *This group homomorphism is surjective, i.e., every automorphism of* $\mathbb{D}$ *can be obtained in this way.*

Proof. If C is the matrix of Exercise 51, then $C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}$ and $M_C^{-1} = M_{C^{-1}}$. It is easy to verify by matrix multiplication that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the indicated group homomorphism yields the automorphism $\mathbb{D} \to \mathbb{D}$ given by

$$z \mapsto \frac{xz + y}{\overline{y}z + \overline{x}},$$

where $x := a+d+\mathrm{i}b-\mathrm{i}c$ and $y := a-d-\mathrm{i}b-\mathrm{i}c$. If a, b, c, d were arbitrary real numbers, x, y would be arbitrary complex numbers. It is easy to verify that $|x|^2 - |y|^2 = 4 \det A$. Thus, the condition $\det A > 0$ on (a, b, c, d) corresponds precisely to the condition $|x| > |y|$ on (x, y). Recalling Exercise 47, we recognize that we obtain all automorphisms of $\mathbb{D}$. $\square$

Exercise 52. Let $A, B \in \mathrm{GL}_2(\mathbb{C})$. Show that $M_A = M_B$ iff there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $B = \lambda A$.

$\mathrm{PGL}_2(\mathbb{C})$ is the group of *projective general linear transformations* of $\mathbb{C}^2$. It is the quotient $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$, where $\mathbb{C}^*$ is the subgroup of $\mathrm{GL}_2(\mathbb{C})$ given by non-zero complex multiples of the unit matrix.

Exercise 53. Show that $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to $\mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{SL}_2(\mathbb{C})$ consisting of $\{\mathbf{1}, -\mathbf{1}\}$.

Proposition 4.18. $\mathrm{PGL}_2(\mathbb{C}) \approx$ ***Möb***.

$\mathrm{PGL}_2^+(\mathbb{R})$ is the group of *projective orientation-preserving general linear transformations* of $\mathbb{R}^2$. It is the quotient $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^*$, where $\mathbb{R}^*$ is the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ given by non-zero real multiples of the unit matrix.

Exercise 54. Show that $\mathrm{PGL}_2^+(\mathbb{R})$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ consisting of $\{\mathbf{1}, -\mathbf{1}\}$.

Proposition 4.19. $\mathrm{PGL}_2^+(\mathbb{R}) \approx \mathrm{Aut}(\mathbb{H}) \approx \mathrm{Aut}(\mathbb{D})$.

4.5 Holomorphic Logarithms and Roots

Definition 4.20. A region $D \subseteq \mathbb{C}$ is called *homologically simply connected* iff all holomorphic functions in D are integrable.

Remark 4.21. Theorem 2.43 together with Proposition 2.11 imply that all holomorphic functions are integrable in a region $D \subseteq \mathbb{C}$ iff every closed path γ in D satisfies $\mathrm{Int}_\gamma \subset D$. So this provides an alternative definition of homologically simple connectedness. In fact it turns out that the adjective "homologically" is superfluous as the notion is equivalent to simple connectedness. However, we will not prove this here.

Definition 4.22. Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, $g \in \mathcal{O}(D)$ is called a *holomorphic logarithm* of f iff $f = \exp g$.

Theorem 4.23. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $f \in \mathcal{O}(D)$ zero-free. Then, there exists a holomorphic logarithm of f in D.*

Proof. By the assumptions $f'/f \in \mathcal{O}(D)$ and integrable. Let $h \in \mathcal{O}(D)$ be a primitive. Define $k := f \exp(-h) \in \mathcal{O}(D)$. As is easy to check, $k' = 0$ so $k = c$ for all $z \in D$ for some constant $c \in \mathbb{C}$. This implies $f = c \exp h$ and $c \neq 0$ since f is zero-free. Since exp takes all complex values except zero, there is $b \in \mathbb{C}$ with $c = \exp(b)$. Then, $g := h + b \in \mathcal{O}(D)$ is the looked for holomorphic logarithm with $f = \exp g$. $\square$

Definition 4.24. Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ and $n \in \mathbb{N}$. Then, a *(holomorphic) nth root* of f is a function $g \in \mathcal{O}(D)$ such that $f = g^n$.

Theorem 4.25. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $f \in \mathcal{O}(D)$ zero-free. Then, there exists an nth root of f for every $n \in \mathbb{N}$.*

Proof. According to Theorem 4.23 there is a holomorphic logarithm $g \in \mathcal{O}(D)$ of f. An nth root of f is given by

$$z \mapsto \exp\left(\frac{1}{n} g\right) \quad \forall z \in D.$$

$\square$

Exercise 55. Let $D, D' \subseteq \mathbb{C}$ be homologically simply connected regions. Suppose that $D'' := D \cap D'$ is connected and non-empty. Show that D'' is homologically simply connected.

Exercise 56. Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that f is not constant. Let $a \in D$. Show the equivalence of the following statements:

1. There exists a neighborhood $U \subseteq D$ of a such that f has a holomorphic square-root in U.
2. $f(a) \neq 0$ or $f(a) = 0$ and the order of the zero is even.

Exercise 57. Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that f is not constant. Show that for any $a \in D$ there exists a neighborhood $U \subseteq D$ of a such that there is $m \in \mathbb{N}$ and $g \in \mathcal{O}(U)$ biholomorphic with the property $f(z) = f(a) + (g(z))^m$ for all $z \in U$.